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# Mathematical evaluation of the synchronization of mechanical oscillations in a cow milking system

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## Abstract

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Organized mechanical vibrations in the frequency range of several times of tens of Hz in the milk production elements of cows ensure health and essentially stop the development of hazardous processes in them. Electromechanical vibrator driven by unbalanced DC electric motor of asynchronous type is attached to each teat cup of the milking machine milk producing element. A mathematical model of the system is obtained, which enables us to observe the vibrating processes analytically – graphically as well as to predict the dynamic synchronization of those vibrations. The purpose of the work is to create organized vibrating processes and to improve the health of animals as well as the quality of milking process.

*Keywords:* process of producing milk of cows; forced mechanical vibrations; forced frequencies; dynamic synchronization through the tissues of milk producing elements

## Introduction

Proper stimulation of the cow‘s teats and udder can enhance milk ejection, which is required for completeness of milk removal at milking. In this work modification of the milking machine is presented, such as application of the teat cup vibrations to stimulate the cow‘s udder during the milking process. This can improve milking out the alveolar milk (high up in the udder) with the cisternal milk (in the udder just above the teat), and cows will milk out cleaner and faster. It contributes that cow milk faster, gives more milk with better milk quality and will help reduce the risk of mastitis.

This research presents the influence of the non-linear phenomena on the behavior of teats cups vibration driven by four unbalanced DC electric motors. Based on a simplified model with 3 degrees of freedom previously published by the authors (Sederevičius et al., 2023), we investigate the dynamics of self-synchronization of rotating shafts during the milking process, in which the teat cup vibration is achieved and when four DC motors have limited power supplies (non-ideal) and with masses attached eccentrically to their rotating shafts.

Effects of nonlinear vibrations are investigated in study of Blekhman (2018). Mechanisms of robots are analyzed in research of Glazunov (2018). Synchronization of exciters of vibrations is investigated by Kibirkštis et al. (2017, 2018). Vibrations of transmissions are presented in research of Kurila and Ragulskienė (1986). Essentially nonlinear vibrating systems are analyzed from Ragulskienė (1974). Synchronizations in vibrating systems are investigated in research of Ragulskis et al. (1965). Precise robots are described in a study of Ragulskis et al. (1987).

An ideal energy source is one that acts on the vibrating system, but does not experience any influence from the system, and a non-ideal source is one that acts on a vibrating system, and at the same time experiences a reciprocal action from the system. Alteration in the parameters of the system may be accomplished by alterations in the working conditions of the energy source. These interactions may become especially active when the energy source has very limited power and in regions of resonance. Here, we are interested in analyzing the influence of the response of the elastic udder on the transient self-synchronization of the DC motors.

Self-synchronization and synchronization will be analyzed by considering the velocity and phase differences in the responses graphically obtained via MatlabTM.

In this article, the vacuum-assisted milking system stimulated by the harmonic low-frequency vibrations was investigated.

Research hypothesis is that the self-synchronization of the unbalanced DC motors can be used to improve the massaging effect on the mammary gland and milking performance and minimize the negative mechanical effect on the teat tissue.

## Mathematical model of the vibrating system

As shown in Fig. 1, the system under investigation consists of a structural model with lumped masses. The dynamic model of the vibrating system is with four unbalanced rotors, attached to a main rigid frame 1, which is supported by spring-damper system on the ground. It is assumed that the motion of the total system is the plane motion. The mass of the non-rotating parts of the milking system which consists of the teat cup, hoses and stator of the DC motor is $m_0$, and the unbalanced rotating part has mass $m_i$ and rotary inertia $I_i$ about the spin axis ($z$-axis). The center of mass of the rotating part is at $B_i$, which is at a distance $r_i$ from the geometric center $A_i$ lying on the rotating shaft axis. The stiffness and damping coefficients of the foundation are denoted by $C$ and $H$, respectively, and $(x, y)$ defines the inertial reference coordinates. The unbalanced rotor spin angle about the geometric center is $\varphi_i$ with phase $\beta_i$.

The vacuum-assisted milking system activated by the harmonic low-frequency vibrations is shown in Fig. 2, oscillates in lateral plane defined by two orthogonal directions $x$ and $y$, with four exciters of vibration attached to the teat cups.

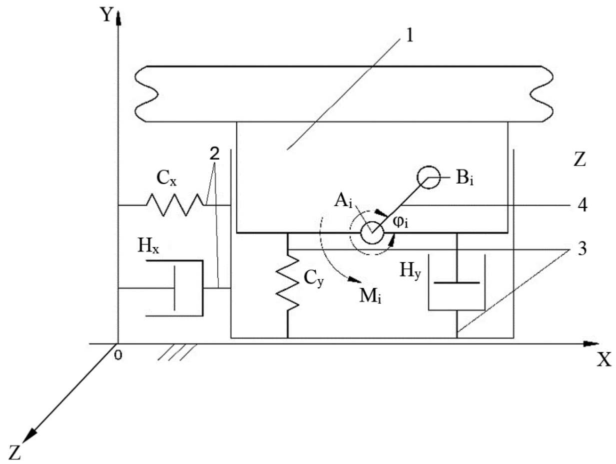


**Fig. 1. Dynamic model of the vibrating system: 1 is a mass $m_0$ vibrating in the directions of $OX$ and $OY$; 2 and 3 are the connecting elements of the mass 1, where $C_x$ and $C_y$, $H_x$ and $H_y$ – elements of linear elastic stiffness and viscous friction; 4 is the rotor of the vibrator with the eccentric mass $m_i$ located at the point $B_i$ and with the moment of inertia $I_i$ with respect to the point $A_i$; $A_i(x, y)$, $B_i(x_B = x + r_i\cos\varphi_i, y_B = y + r_i\sin\varphi_i)$, $r_i = A_iB_i$, $M_i$ is the moment of rotation of the rotor $i$**

*Source*: Authors' own elaboration

The experiment uses an artificial model of a cow's udder, with 4 milkers, and a vibrator attached to each of them. The resonant frequencies of the vibrators are different from each other, due to the different physical parameters of the teats in

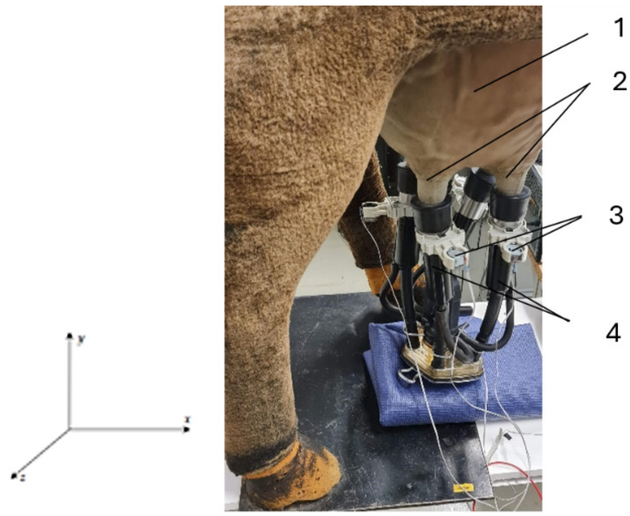


**Fig. 2. The milking system, experimental model: 1 – artificial cow udder model, 2 – teats of the model, 3 – unbalanced vibratory motor, 4 – milker device**

*Source*: Authors' own elaboration

the circular sections of the udder. It is important to align (synchronize) these vibrators with each other, in order to achieve the optimum operating mode during the milking process.

Kinetic and potential energies, dissipative function of the system and external moments of rotational forces of rotors, when $y$ axis coincides with the force of gravity, are:

$$
\begin{aligned}
&T=\frac{1}{2}\left(m_0+\sum_{i=1}^{4} m_i\right)\left(\dot{x}^2+\dot{y}^2\right)+\sum_{i=1}^{4} m_i r_i \dot{\phi}_i\left(-\dot{x}\sin\varphi_i+\dot{y}\cos\varphi_i\right)+\frac{1}{2}\left(I_i+mr_i^2\right)\dot{\phi}_i^2,\\
&\Pi=\frac{1}{2}\left(C_x x^2+C_y y^2\right)+g\sum_{i=1}^{4} m_i\left(y-r_i\sin\varphi_i\right),\\
&D=\frac{1}{2}\left(H_x\dot{x}^2+H_y\dot{y}^2\right),\\
&M_i=M_i\left(\dot{\phi}_i\right).
\end{aligned}
\tag{1}
$$

Differential equations of motion are:

$$
\begin{aligned}
&\left(m_0+\sum_{i=1}^{4} m_i\right)\ddot{x}-\sum_{i=1}^{4} m_i r_i\left(\ddot{\phi}_i\sin\varphi_i+\dot{\phi}_i^2\cos\varphi_i\right)+C_x x+H_x\dot{x}=0,\\
&\left(m_0+\sum_{i=1}^{4} m_i\right)\ddot{y}+\sum_{i=1}^{4} m_i r_i\left(\ddot{\phi}_i\cos\varphi_i-\dot{\phi}_1^2\sin\varphi_i\right)+g\sum_{i=1}^{4} m_i+C_y y+H_y\dot{y}=0,\\
&\left(I_i+m_i r_i^2\right)\ddot{\phi}_i+m_i r_i\left(-\ddot{x}\sin\varphi_i+\ddot{y}\cos\varphi_i\right)-g m_i r_i\cos\varphi_i=M_i\left(\dot{\phi}_i\right),
\end{aligned}
\tag{2}
$$

or:

$$
\ddot{x}+p_x^2 x+h_x\dot{x}=+\sum_{i=1}^{4}\mu_i r_i\left(\ddot{\varphi}_i\sin\varphi_i+\dot{\varphi}_i^2\cos\varphi_i\right),
\tag{3}
$$

$$
\ddot{y}+p_y^2 y+h_y\dot{y}+g\sum_{i=1}^{4}\mu_i=-\sum_{i=1}^{4}\mu_i r_i\left(\ddot{\varphi}_i\cos\varphi_i-\dot{\varphi}_i^2\sin\varphi_i\right),
\tag{4}
$$

$$
\ddot{\varphi}_i+\frac{\nu_i}{r_i}\left(-\ddot{x}\sin\varphi_i+\ddot{y}\cos\varphi_i\right)-g\frac{\nu_i}{r_i}\cos\varphi_i-A_i+B_i\dot{\varphi}_i=0,
\tag{5}
$$

where:

$$
\begin{aligned}
&\mu_i=\frac{m_i}{m_0+\sum_{i=1}^{4} m_i},\ p_x^2=\frac{C_x}{m_0+\sum_{i=1}^{4} m_i},\ h_x=\frac{H_x}{m_0+\sum_{i=1}^{4} m_i},\ p_y^2=\frac{C_y}{m_0+\sum_{i=1}^{4} m_i},\\
&h_y=\frac{H_y}{m_0+\sum_{i=1}^{4} m_i},\ \nu_i=\frac{m_i r_i^2}{I_i+m_i r_i^2},\ A_i=\frac{M_i\left(\dot{\varphi}_i\right)\big|_{\dot{\varphi}_i=0}}{I_i+m_i r_i^2},\ B_i=\frac{M_i\left(\dot{\varphi}_i\right)\big|_{\dot{\varphi}_i=0}}{I_i+m_i r_i^2}.
\end{aligned}
\tag{6}
$$

By considering that in the analyzed case:

$$\dot{\varphi}_i = \omega t + \beta_i,$$
$$\ddot{\varphi}_i \ll \dot{\varphi}_i^2, \tag{7}$$
$$\dddot{\varphi} = g = 0,$$

the equations (3) – (5) and by considering (7) take the form:

$$\ddot{x} + h_x \dot{x} + p_x^2 x = \omega^2 \sum_{i=1}^{4} \mu_i r_i \cos(\omega t + \beta_i), \tag{8}$$

$$\ddot{y} + h_y \dot{y} + p_y^2 y = \omega^2 \sum_{i=1}^{4} \mu_i r_i \sin(\omega t + \beta_i), \tag{9}$$

$$\ddot{\varphi}_i - \frac{\nu_i}{r_i}(\ddot{x} \sin\varphi_i + \ddot{y} \cos\varphi_i) + A_i - B_i \dot{\varphi}_i = 0. \tag{10}$$

Steady state motion of the equation (8) is:

$$x = \sum_{i=1}^{4} \left[ R_i \cos(\omega t + \beta_i) + S_i \sin(\omega t + \beta_i) \right], \tag{11}$$

$$\sum_{i=1}^{4} R_i = -\frac{1 - \left(\frac{p_x}{\omega}\right)^2}{\left[1 - \left(\frac{p_x}{\omega}\right)^2\right]^2 + \left(\frac{h_x}{\omega}\right)^2} \sum_{i=1}^{4} \mu_i r_i, \tag{12}$$

$$\sum_{i=1}^{4} S_i = -\frac{\frac{h_x}{\omega}}{\left(\frac{p_x}{\omega}\right)^2} \sum_{i=1}^{4} R_i = \left[ 1 - \frac{\frac{h_x}{\omega}}{1 - \left(\frac{p_x}{\omega}\right)^2} \sum_{i=1}^{4} R_i \right], \tag{13}$$

$$\ddot{\varphi}_i - A_i + B\dot{\varphi}_i - \frac{\nu_i}{r_i} \ddot{x} \sin\varphi_i = 0, \tag{14}$$

$$\ddot{\beta}_i - A_i + B\omega + B\dot{\beta}_i + \frac{\nu_i}{r_i}\omega^2 \left[ R_x \cos(\omega t + \beta_i) + S_x \sin(\omega t + \beta_i) \right] \sum_{i=1}^{4} \sin(\omega t + \beta_i), \tag{15}$$

where:

$$R_x = -\frac{1 - \left(\frac{p_x}{\omega}\right)^2}{\left[1 - \left(\frac{p_x}{\omega}\right)^2\right] - \left(\frac{h_x}{\omega}\right)^2}, \tag{16}$$

$$S_x = \frac{\frac{h_x}{\omega}}{1-\left(\frac{p_x}{\omega}\right)^2} R, \tag{17}$$

$$\ddot{\beta}_i - A_i + B\omega + B\dot{\beta}_i + \\ +\omega^2 \frac{1}{1-\left(\frac{h_x}{\omega}\right)^2}\left[\cos(\omega t+\beta_i) - \frac{\left(\frac{h_x}{\omega}\right)^2}{1-\left(\frac{h_x}{\omega}\right)^2}\sin(\omega t+\beta_i)\right]\sum_{i=1}^{4}\mu_i r_i(\omega t+\beta_i) = 0. \tag{18}$$

## Model of the rotary system with one degree of freedom

The system with one degree of freedom driven by a one unbalanced DC motor is described by the following equation:

$$I\ddot{\varphi} + H\dot{\varphi} = M(\dot{\varphi}), \tag{19}$$

where $I$ is the moment of inertia, $\varphi$ is the angle of rotation, $H$ is the coefficient of viscous damping, $M$ is the driving moment of the DC motor and the upper dot indicates differentiation with respect to time.

This equation is rearranged as:

$$\ddot{\varphi} + h\dot{\varphi} = \frac{M(\dot{\phi})}{I}, \tag{20}$$

where:

$$h = \frac{H}{I}. \tag{21}$$

For the DC motor it is assumed that:

$$\frac{M(\dot{\phi})}{I} = A - B\dot{\varphi}, \tag{22}$$

where $A$ and $B$ are constants determining the behavior of the DC motor.

It is assumed that:

$$\varphi = \omega t + \beta, \tag{23}$$

where $\omega$ is the angular velocity of rotation, $t$ is the time variable and $\beta$ is the phase.

Thus, it is obtained:

$$\dot{\varphi} = \omega \div \dot{\beta}. \tag{24}$$

Dynamics of the investigated system is described by the following equation:

$$\ddot{\varphi} + h\dot{\varphi} = A - B\dot{\varphi}. \tag{25}$$

It follows that:

$$\ddot{\beta} \div h\omega \div h\dot{\beta} = A - B\omega - B\dot{\beta}. \tag{26}$$

Thus:

$$\ddot{\beta} + \left(h + B\right)\dot{\beta} = A - B\omega - h\omega. \tag{27}$$

Angular frequency of rotation is determined from the following equation:

$$A - B\omega - h\omega = 0. \tag{28}$$

Thus:

$$\omega = \frac{A}{B + h}. \tag{29}$$

## Model of the system with four degrees of freedom

The system is described by the following equations:

$$\begin{cases} \ddot{\beta}_1 + h\omega + h\dot{\beta}_1 + B\dot{\beta}_1 - \left(A_1 - B\omega\right) + \\ + \frac{1}{2}ab\omega^2\left[\sin\left(\beta_1 - \beta_2\right) + \sin\left(\beta_1 - \beta_3\right) + \sin\left(\beta_1 - \beta_4\right)\right] = 0, \\ \ddot{\beta}_2 + h\omega + h\dot{\beta}_2 + B\dot{\beta}_2 - \left(A_2 - B\omega\right) + \\ + \frac{1}{2}ab\omega^2\left[\sin\left(\beta_2 - \beta_1\right) + \sin\left(\beta_2 - \beta_3\right) + \sin\left(\beta_2 - \beta_4\right)\right] = 0, \\ \ddot{\beta}_3 + h\omega + h\dot{\beta}_3 + B\dot{\beta}_3 - \left(A_3 - B\omega\right) + \\ + \frac{1}{2}ab\omega^2\left[\sin\left(\beta_3 - \beta_1\right) + \sin\left(\beta_3 - \beta_2\right) + \sin\left(\beta_3 - \beta_4\right)\right] = 0, \\ \ddot{\beta}_4 + h\omega + h\dot{\beta}_4 + B\dot{\beta}_4 - \left(A_4 - B\omega\right) + \\ + \frac{1}{2}ab\omega^2\left[\sin\left(\beta_4 - \beta_1\right) + \sin\left(\beta_4 - \beta_2\right) + \sin\left(\beta_4 - \beta_3\right)\right] = 0, \end{cases} \tag{30}$$

where $\beta_1$, $\beta_2$, $\beta_3$, $\beta_4$ are the phases of the four degrees of freedom, $h$ is the coefficient of viscous damping, $A_1$, $A_2$, $A_3$, $A_4$

are parameters of the four DC motors and $B$ is the parameter of the four DC motors, $\omega$ is the angular velocity of rotation, $a$ and $b$ are constant parameters of the investigated system.

The system can be rewritten as:

$$\begin{cases} \ddot{\beta}_1 + h\omega + h\dot{\beta}_1 + B\dot{\beta}_1 - (A_1 - B\omega) + \\ + \frac{1}{2}ab\omega^2\left[\sin(\beta_1-\beta_2)+\sin(\beta_1-\beta_3)+\sin(\beta_1-\beta_4)\right]=0, \\ \ddot{\beta}_2 + h\omega + h\dot{\beta}_2 + B\dot{\beta}_2 - (A_2 - B\omega) + \\ + \frac{1}{2}ab\omega^2\left[-\sin(\beta_1-\beta_2)+\sin(\beta_2-\beta_3)+\sin(\beta_2-\beta_4)\right]=0, \\ \ddot{\beta}_3 + h\omega + h\dot{\beta}_3 + B\dot{\beta}_3 - (A_3 - B\omega) + \\ + \frac{1}{2}ab\omega^2\left[-\sin(\beta_1-\beta_3)-\sin(\beta_2-\beta_3)+\sin(\beta_3-\beta_4)\right]=0, \\ \ddot{\beta}_4 + h\omega + h\dot{\beta}_4 + B\dot{\beta}_4 - (A_4 - B\omega) + \\ + \frac{1}{2}ab\omega^2\left[-\sin(\beta_1-\beta_4)-\sin(\beta_2-\beta_4)-\sin(\beta_3-\beta_4)\right]=0. \end{cases} \quad (31)$$

Initial value of $\omega$ is determined by solving the following equations:

$$\begin{cases} h\omega - (A_1 - B\omega) = 0, \\ h\omega - (A_2 - B\omega) = 0, \\ h\omega - (A_3 - B\omega) = 0, \\ h\omega - (A_4 - B\omega) = 0. \end{cases} \quad (32)$$

Thus, four values of the frequency are determined:

$$\begin{cases} \omega_1 = \frac{A_1}{h+B}, \\ \omega_2 = \frac{A_2}{h+B}, \\ \omega_3 = \frac{A_3}{h+B}, \\ \omega_4 = \frac{A_4}{h+B}. \end{cases} \quad (33)$$

Then the average frequency is calculated as:

$$\omega_0 = \frac{1}{4}\sum_{i=1}^{4}\omega_i = \frac{1}{h+B}\cdot\frac{A_1+A_2+A_3+A_4}{4}. \quad (34)$$

From the previous equations it is determined:

$$\dot{\beta}_i = \dot{\bar{\beta}}_i + \dot{\tilde{\beta}}_i, \quad (35)$$

where:

$$\bar{\dot{\beta}}_i = const, \tag{36}$$

and:

$$\bar{\ddot{\beta}}_i = 0. \tag{37}$$

Then the following quantities are determined:

$$\begin{cases} \omega_{10} = \omega_1 + \bar{\dot{\beta}}_1, \\ \omega_{20} = \omega_2 + \bar{\dot{\beta}}_2, \\ \omega_{30} = \omega_3 + \bar{\dot{\beta}}_3, \\ \omega_{40} = \omega_4 + \bar{\dot{\beta}}_4. \end{cases} \tag{38}$$

When those four frequencies are mutually equal, then the system performs synchronous motions. That is frequencies of all exciters of vibrations are mutually equal.

If those four frequencies are mutually not equal, then their average value is found, that is:

$$\omega_{01} = \frac{1}{4}\sum_{i=1}^{4} \omega_{i0}, \tag{39}$$

This value is used instead of $\omega$ in the previous equations.

## Model of the rotary system with four degrees of freedom driven by two unbalanced DC motors

Differential equations of motion are:

$$\begin{aligned} &\left(m_0 + \sum_{i=1}^{2} m_i\right)\ddot{x} - \sum_{i=1}^{2} m_i r_i\left(\ddot{\varphi}_i \sin\varphi_i + \dot{\varphi}_i^2 \cos\varphi_i\right) + C_x x + H_x \dot{x} = 0, \\ &\left(m_0 + \sum_{i=1}^{2} m_i\right)\ddot{y} + \sum_{i=1}^{2} m_i r_i\left(\ddot{\varphi}_i \cos\varphi_i - \dot{\varphi}_i^2 \sin\varphi_i\right) + C_y y + H_y \dot{y} = 0, \\ &\left(I_i + m_i r_i^2\right)\ddot{\varphi}_i + m_i r_i\left(-\ddot{x}\sin\varphi_i + \ddot{y}\cos\varphi_i\right) + B_i \dot{\varphi}_i = A_i,\ i = 1,2. \end{aligned} \tag{40}$$

The following values of the parameters of the investigated system are assumed:

$$\begin{aligned} &m_0 = 1,\ m_1 = 0.1,\ m_2 = 0.1,\ r_1 = 1,\ r_2 = 1,\ H_x = 0.1,\ C_x = 1,\ H_y = 0.1,\ C_y = 1, \\ &I_1 = 1,\ B_1 = 0.1,\ I_2 = 1,\ B_2 = 0.1. \end{aligned} \tag{41}$$

First it is assumed that both driving moments are mutually equal:

$$A_1 = 0.1,\ A_2 = 0.1. \tag{42}$$

Results of investigations are presented in Fig. 3.

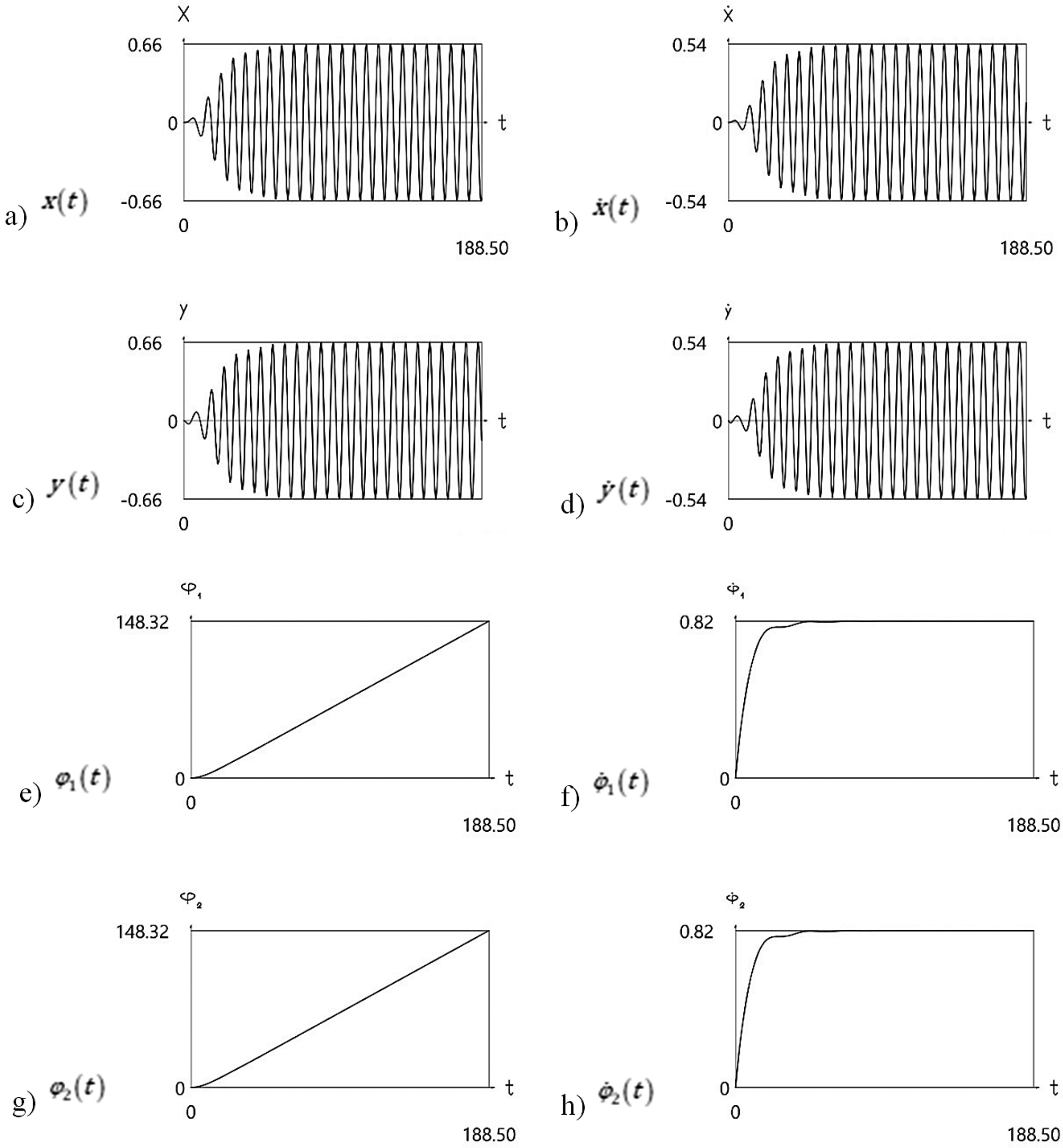


**Fig. 3. Dynamics of the system during the starting process when driving moments of the two exciters of vibrations are mutually equal: a, b – displacement and velocity of the frame with DC motors in *x* direction; c, d – displacement and velocity of the frame with DC motors in *y* direction; e, f – angular displacement and velocity of the first rotor; g, h – angular displacement and velocity of the second rotor**

*Source*: Authors' own elaboration

From the presented results it is seen that both exciters of vibrations during the starting process move in the same way and steady state motion is reached after 7 periods.

Then it is assumed that the driving moments differ by a small amount:

$$A_1 = 0.09,\ A_2 = 0.11. \quad (43)$$

Results of investigations are presented in Fig. 4.

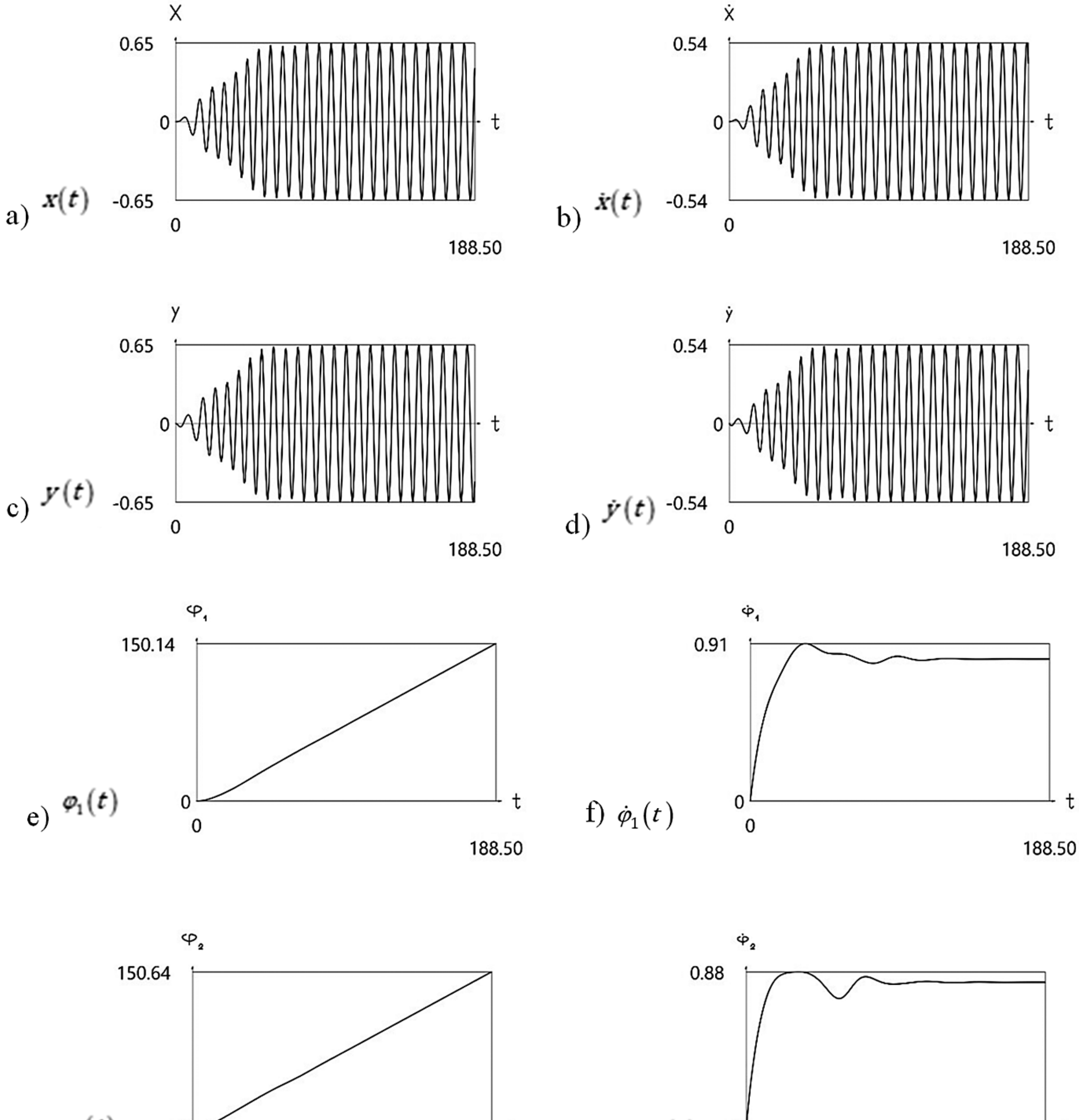


**Fig. 4. Dynamics of the system during the starting process when driving moments of the two exciters of vibrations differ by a small amount: a, b – displacement and velocity of the frame with DC motors in *x* direction; c, d – displacement and velocity of the frame with DC motors in *y* direction; e, f – angular displacement and velocity of the first rotor; g, h – angular displacement and velocity of the second rotor**

*Source*: Authors' own elaboration

Both angular velocities of the two exciters of vibrations are presented in Fig. 5.

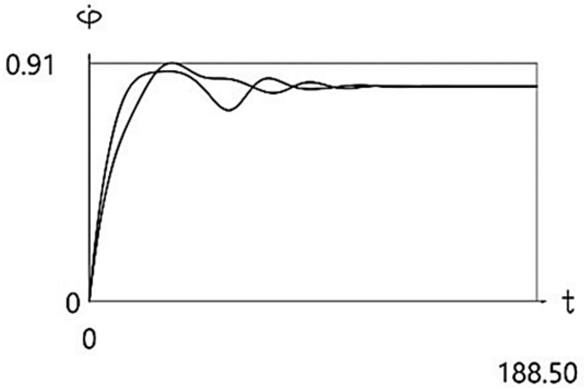


**Fig. 5. The angular velocities of the two exciters of vibrations when driving moments differ by a small amount**

*Source*: Authors' own elaboration

From the presented results the effect of synchronization is seen after some 10 periods of oscillation.

## Investigation of the system with four exciters of vibrations

Differential equations of motion are:

$$\left(m_0+\sum_{i=1}^{4}m_i\right)\ddot{x}-\sum_{i=1}^{4}m_i r_i\left(\ddot{\varphi}_i\sin\varphi_i+\dot{\varphi}_i^2\cos\varphi_i\right)+C_x x+H_x\dot{x}=0,$$
$$\left(m_0+\sum_{i=1}^{4}m_i\right)\ddot{y}+\sum_{i=1}^{4}m_i r_i\left(\ddot{\varphi}_i\cos\varphi_i-\dot{\varphi}_i^2\sin\varphi_i\right)+C_y y+H_y\dot{y}=0, \qquad (44)$$
$$\left(I_i+m_i r_i^2\right)\ddot{\varphi}_i+m_i r_i\left(-\ddot{x}\sin\varphi_i+\ddot{y}\cos\varphi_i\right)+B_i\dot{\varphi}_i=A_i,\ i=1,2,3,4.$$

The following values of the parameters of the investigated system are assumed:

$$m_0=1,\ m_1=0.1,\ m_2=0.1,\ m_3=0.1,\ m_4=0.1,\ r_1=1,\ r_2=1,\ r_3=1,\ r_4=1,\ H_x=0.1,$$
$$C_x=1, H_y=0.1, C_y=1, I_1=1, I_2=1,\ I_3=1,\ I_4=1,\ B_1=0.1, B_2=0.1, B_3=0.1, B_4=0.1. \qquad (45)$$

It is assumed that the DC motors are not identical, and their driving moments differ by a small amount:

$$A_1=0.09,\ A_2=0.11,\ A_3=0.09,\ A_4=0.11. \qquad (46)$$

Results of investigations are presented in Fig. 6.

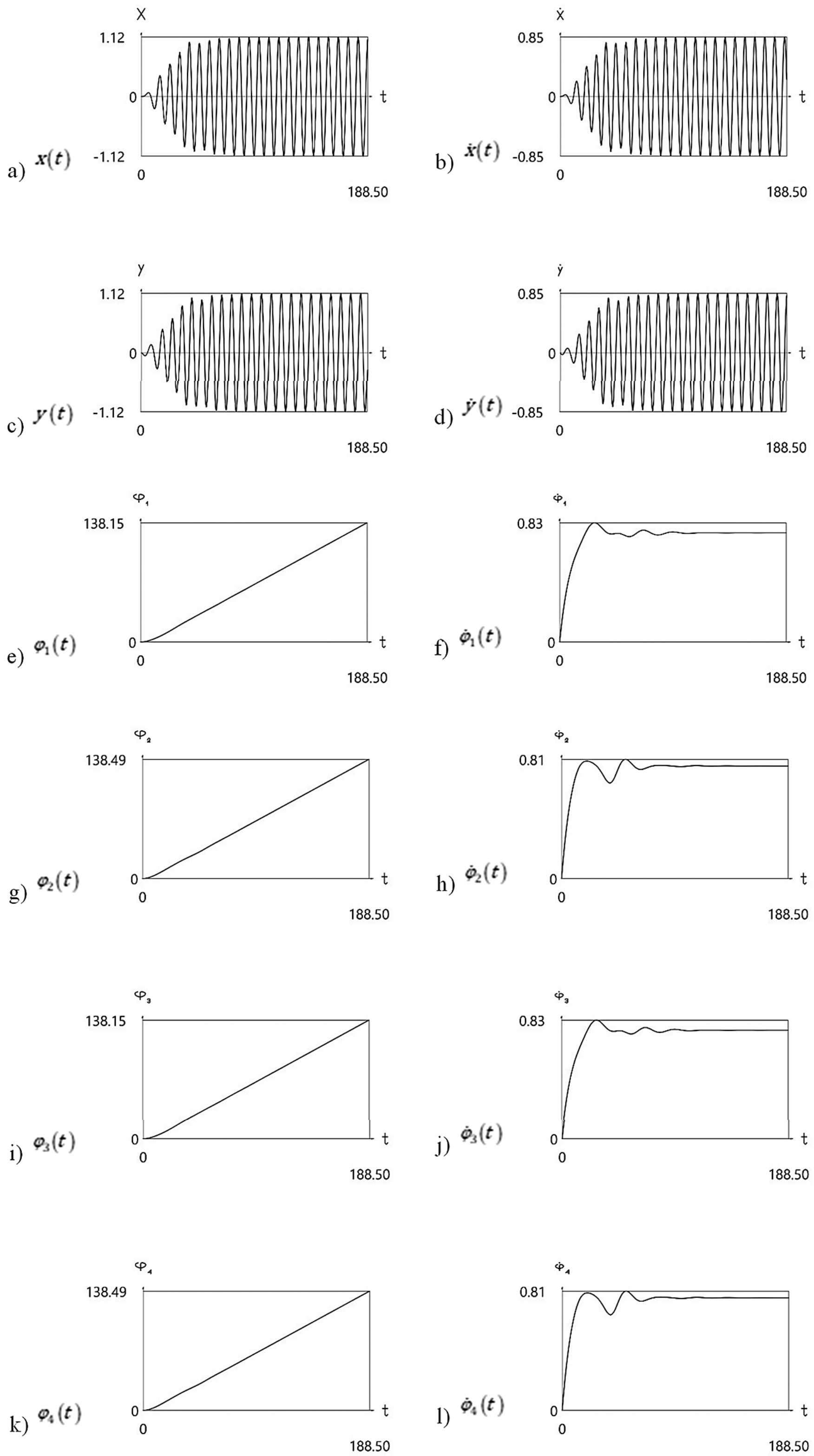


**Fig. 6. Dynamics of the system during the starting process when driving moments of the four exciters of vibrations differ by a small amount: a, b – displacement and velocity of the frame with DC motors in *x*; c, d – displacement and velocity of the frame with DC motors in *y* directions; e, f – angular displacement and velocity of the first rotor; g, h – angular displacement and velocity of the second rotor; i, j – of the third rotor; k, l – of the fourth rotor**

*Source*: Authors' own elaboration

All angular velocities are presented in Fig. 7.

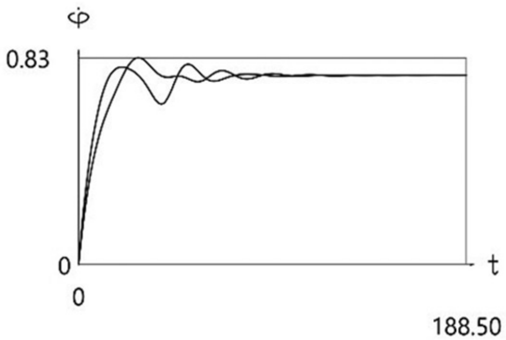


**Fig. 7. The angular velocities of the four exciters of vibrations when their driving moments differ by a small amount**

*Source*: Authors' own elaboration

From the presented results the effect of synchronization is seen after some 9 periods of oscillation.

## Conclusions

The modification of the milking machine is presented, where the teat cup vibrations are driven by four unbalanced DC motors to stimulate the cow's udder during the milking process.

A dynamic model of the vibrating system with unbalanced DC motors is described, and equations of motion are obtained. In the model it is assumed that there are four exciters of vibrations driven by DC motors.

A non-linear phenomenon of self-synchronization in pre-resonance region between a number of unbalanced DC motors, interacting with their flexible structural frame foundation and response have been analyzed through numerical simulations.

Numerical investigations of the system show, that self-synchronization effected an increase of vibrations excited by non-ideal vibrators (unbalanced DC motors of limited power). The exciters of vibrations during the starting process move in the same way and steady state motion is reached after some 9 – 11 periods of oscillation.

There are some small fluctuations in time domain for the vibrations in the steady state, generally these fluctuations are so small that they cannot influence the stimulating function of vibrating effect during the milking process.

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